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Meromorphically Starlike Functions at Infinity

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Abstract. In this letter, we introduce meromorphically starlike functions by using a particular class of analytic functions that present authors applied in 2012 [cf.[1]]. These functions map the punctured unit disk into a starlike domain at infinity. Besides, we investigate coefficient estimate, radii of convexity and distortion theorems as well. Finally, we ascertain that these subclasses are closed under convex linear combination.

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1. Introduction

Let $\sum$ stand for the class of

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1. 1}$$

that are analytic in U^*, where $U^* = \{z : 0 < |z| < 1\}$.

$\sum_{\alpha}$ represent all of $f(z)$ such that

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{nq}, q = 1 - \frac{1}{\alpha}, \alpha \in \mathbf{N}, \tag{1. 2}$$

and are analytic in U^*. When α goes to infinity then $1 - 1/\alpha$ approaches to 1; hence $\sum_{\alpha} = \sum$.
$\sum_{\alpha}^{+}$ also denote functions such as

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n| z^{nq}, q = 1 - \frac{1}{\alpha}, \alpha \in \mathbf{N}, \tag{1. 3}$$

and analytic in U^*.
In view of above definitions, We define a generalized subclass, denoted by $\sum_{\alpha}(\beta, \delta, \gamma)$ for f belonging to $\sum_{\alpha}$ as follows.
A function f defined by (1. 2) belongs to $\sum_{\alpha}(\beta, \delta, \gamma)$, if it get by analytic criterion s.t.

$$\left| \frac{\frac{zf'(z)}{f(z)} + 1}{2\gamma \left[\frac{zf'(z)}{f(z)} + \beta \right] - \left[\frac{zf'(z)}{f(z)} + 1 \right]} \right| < \delta, z \in U^*, \tag{1. 4}$$

for $0 \leq \beta < 1, 0 < \delta \leq 1$ and $\frac{1}{2} < \gamma \leq 1$. When $\alpha \to \infty$ then $1 - 1/\alpha \to 1$; hence $\sum_{\alpha} = \sum$ implies $\sum_{\alpha}(\beta, \delta, \gamma)$ equal to $\sum(\beta, \delta, \gamma)$.
We consider $\sum_{\alpha}^{+}(\beta, \delta, \gamma)$ equal to $\sum_{\alpha}^{+} \cap \sum_{\alpha}(\beta, \delta, \gamma)$, clearly $\sum_{\alpha}^{+}(\beta, \delta, \gamma)$ is the generalized form of the class $\sum_{\alpha}(\beta, \delta, \gamma)$ that Aouf and Joshi introduced in 1998 [cf.[2]]. Similar work has been seen for different subclasses done by other author's[see for example [3-5]].
Using (1. 4), we obtain the characterization properties for the special family of meromorphic functions at infinity defined in (1. 2) as follow:

Theorem 1.1. *A $f(z) \in \sum_{\alpha}^{+}$ belongs to $\sum_{\alpha}^{+}(\beta, \delta, \gamma)$, iff it satisfies the analytic criterion*

$$\sum_{n=1}^{\infty} \{(nq + 1) + \delta[(1 - nq) + 2\gamma(nq - \beta)]\}|a_n| \leq 2\delta\gamma(1 - \beta), \tag{1. 5}$$

where $0 \leq \beta < 1, 0 < \delta \leq 1, \frac{1}{2} < \gamma \leq 1$ and q having the same constraints as given in (1. 2).

Proof. Let $f \in \sum_{\alpha}^{+}(\beta, \delta, \gamma)$, then by using (1. 4) and after some calculation, we have

$$\Re \left(\frac{\sum_{n=1}^{\infty} (nq+1)|a_n| z^{nq+1}}{2\gamma(1-\beta) - \sum_{n=1}^{\infty} (1-2\gamma) nq |a_n| z^{nq+1} - \sum_{n=1}^{\infty} (1-2\gamma\beta)|a_n| z^{nq+1}} \right) < \delta, z \in U^*.$$

Letting $z \to 1$ through positive values and after doing some mathematics, we obtain

$$\sum_{n=1}^{\infty} \{(nq + 1) + \delta[(1 - nq) + 2\gamma(nq - \beta)]\}|a_n| \leq 2\delta\gamma(1 - \beta).$$

Conversely, since we have

$$\left| zf'(z) + f(z)| - \delta|2\gamma(zf'(z) + \beta f(z)) - (zf'(z) + f(z)) \right| =$$

$$\left|\sum_{n=1}^{\infty}(nq+1)a_n z^{nq}\right| - \delta\Big|2\gamma(\beta-1)\tfrac{1}{z} +$$
$$\sum_{n=1}^{\infty}(2\gamma-1)(nq)a_n z^{nq} + \sum_{n=1}^{\infty}(2\beta\gamma-1)a_n z^{nq}\Big|,$$
$$\leq \left|\sum_{n=1}^{\infty}(nq+1)a_n z^{nq}\right| -$$
$$\delta\left[\left|2\gamma(\beta-1)\tfrac{1}{z} + \sum_{n=1}^{\infty}(2\gamma-1)(nq)a_n z^{nq}\right| + \left|\sum_{n=1}^{\infty}(1-2\beta\gamma)a_n z^{nq}\right|\right],$$
or
$$\leq \sum_{n=1}^{\infty}(nq+1)|a_n|r^{nq+1} - \delta\Big\{2\gamma(1-\beta) + \sum_{n=1}^{\infty}(1-2\gamma)(nq)|a_n|r^{nq+1} - \sum_{n=1}^{\infty}(1-$$
$$2\beta\gamma)|a_n|r^{nq+1}\Big\},$$

$$= \sum_{n=1}^{\infty}(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)]|a_n|r^{nq+1} - 2\delta\gamma(1-\beta). \tag{1. 6}$$

Hence by taking limit when $r \to -1$, then we have
$$\leq \sum_{n=1}^{\infty}(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)]|a_n| - 2\delta\gamma(1-\beta) \quad \leq 0.$$

It implies that $f(z) \in \sum_{\alpha}^{+}(\beta,\delta,\gamma)$. □

The solution given in (1. 5) is sharp for
$$f(z) = \frac{1}{z} + \frac{2\delta\gamma(1-\beta)z^{nq}}{(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)]}, \tag{1. 7}$$
where $0 \leq \beta < 1, 0 < \delta \leq 1, \frac{1}{2} < \gamma \leq 1$ and q having the same constraints as given in (1. 2).

Theorem 1.2. *If $f(z) \in \sum_{\alpha}^{+}$ belongs to $\sum_{\alpha}^{+}(\beta,\delta,\gamma)$, then*
$$\frac{1}{|z|} - \frac{\delta\gamma(1-\beta)}{(q+1)+\delta(1-q)+2\delta\gamma(q-\beta)}|z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{\delta\gamma(1-\beta)}{(q+1)+\delta(1-q)+2\delta\gamma(q-\beta)}|z|$$

Proof. Let $f \in \sum_{\alpha}^{+}(\beta,\delta,\gamma)$, then by using our previous Theorem1.1,we have
$$\sum_{n=1}^{\infty}|a_n| \leq \frac{\delta\gamma(1-\beta)}{(q+1) + \delta(1-q) + 2\delta\gamma(q-\beta)},$$
since
$$|f(z)| \leq \frac{1}{|z|} + |z|\sum_{n=1}^{\infty}|a_n|,$$
this implies
$$\leq \frac{1}{|z|} + \frac{\delta\gamma(1-\beta)}{(q+1) + \delta(1-q) + 2\delta\gamma(q-\beta)}|z|.$$
Similarly, one can prove easily that
$$|f(z)| \geq \frac{1}{|z|} - \frac{\delta\gamma(1-\beta)}{(q+1) + \delta(1-q) + 2\delta\gamma(q-\beta)}|z|.$$
□

2. Radii of Convexity

Theorem 2.1. *If $f(z) \in \sum_{\alpha}^{+}$ belongs to $\sum_{\alpha}^{+}(\beta, \delta, \gamma)$, then f is meromorphically convex having order ρ in $|z| < r(\beta, \delta, \gamma, \rho)$, where*

$$r(\beta, \delta, \gamma, \rho) = inf \left[\frac{(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)](1-\rho)}{2\delta\gamma(1-\beta)nq(nq+2-\rho)} \right]^{\frac{1}{nq+1}},$$

where $0 \leq \beta < 1, 0 < \delta \leq 1, \frac{1}{2} < \gamma \leq 1, 0 \leq \rho < 1$ and q having the same constraints as given in (1. 2)*.*

Proof. Suppose $f \in \sum_{\alpha}^{+}(\beta, \delta, \gamma)$, then by Theorem1.1, we get

$$\sum_{n=1}^{\infty} \frac{(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)]}{2\delta\gamma(1-\beta)} |a_n| \leq 1. \tag{2. 8}$$

To prove our main result, it is enough to show that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq (1-\rho),$$

or

$$\left| \frac{f'(z) + (zf'(z))'}{f'(z)} \right| \leq (1-\rho).$$

Constraints on parameters are considered same as given in the statement, so from above expression, we get

$$\left| \frac{\sum\limits_{n=1}^{\infty} nq(nq+1)|a_n|z^{nq-1}}{-\frac{1}{z^2} + \sum\limits_{n=1}^{\infty} nq|a_n|z^{nq-1}} \right| \leq \frac{\sum\limits_{n=1}^{\infty} nq(nq+1)|a_n||z|^{nq+1}}{1 - \sum\limits_{n=1}^{\infty} nq|a_n||z|^{nq+1}}.$$

The expression is bounded above by $(1-\rho)$ if

$$\sum_{n=1}^{\infty} \frac{nq(nq+2-\rho)}{(1-\rho)} |a_n||z|^{nq+1} \leq 1. \tag{2. 9}$$

Using (2. 8), above expression is true if

$$\frac{nq(nq+2-\rho)}{(1-\rho)} |z|^{nq+1} \leq \frac{(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)]}{2\delta\gamma(1-\beta)}, n \in \mathbf{N},$$

impies that

$$|z| \leq \left[\frac{(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)](1-\rho)}{2\delta\gamma(1-\beta)nq(nq+2-\rho)} \right]^{\frac{1}{nq+1}}, n \in \mathbf{N}, \tag{2. 10}$$

and therefore

$$r(\beta, \delta, \gamma, \rho) = inf \left[\frac{(nq+1) + \delta[(1-nq) + 2\gamma(nq-\beta)](1-\rho)}{2\delta\gamma(1-\beta)nq(nq+2-\rho)} \right]^{\frac{1}{nq+1}}, n \in \mathbf{N}.$$

□

3. Extreme Points

Theorem 3.1. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \frac{2\delta\gamma(1-\beta)z^{nq}}{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]}, n \in \mathbf{N},$$

then $f \in \sum_{\alpha}^{+}(\beta,\delta,\gamma) \iff f(z) = \sum_{n=0}^{\infty} \lambda_{nq} f_{nq}(z)$, *where* $\lambda_n \geq 0$ *and* $\sum\limits_{n=0}^{\infty} \lambda_n = 1.$

Proof. Suppose $f(z) = \sum\limits_{n=0}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geq 0$ and $\sum\limits_{n=0}^{\infty} \lambda_n = 1$. Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \lambda_0 f_0(z) + \sum_{n=1}^{\infty} \lambda_n f_n(z)$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n \frac{2\delta\gamma(1-\beta)z^{nq}}{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]}.$$

By using (2. 8) and above expression, we get $\sum_{n=1}^{\infty} \lambda_n = 1-\lambda_0 \leq 1$. Hence by Theorem 1.1, one can see easily that $f \in \sum_{\alpha}^{+}(\beta,\delta,\gamma)$. Conversely, Suppose that $f \in \sum_{\alpha}^{+}(\beta,\delta,\gamma)$, since

$$|a_n| \leq \frac{2\delta\gamma(1-\beta)z^{nq}}{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]}, n \in \mathbf{N}.$$

we adjust

$$\lambda_n = \frac{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]}{2\delta\gamma(1-\beta)}|a_n|, n \in \mathbf{N}$$

and $\lambda_0 = 1-\sum_{n=1}^{\infty} \lambda_n$. Then clearly $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, as required. □

4. Convex linear Combination

Theorem 4.1. $f \in \sum_{\alpha}^{+}(\beta,\delta,\gamma)$ *is closed under convex linear combination.*

Proof. Let $f(z)$ and $g(z)$ belong to $\sum_{\alpha}^{+}(\beta,\delta,\gamma)$, where $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |a_n|z^{nq}$, and $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} |b_n|z^{nq}$. We consider $h(z) = \mu f(z) + (1-\mu)g(z)$, $(0 \leq \mu < 1)$, then by using Theorem 1.1, we have

$$\sum_{n=1}^{\infty} \{(nq+1)+\delta\left[(1-nq)+2\gamma(nq-\beta)\right]\} |\mu a_n + (1-\mu)b_n| \leq 2\delta\gamma(1-\beta).$$

□

Next we discussed a special subclass of meromorphic functions denoted by $\sum_{r,\alpha}^{+}[\beta,\delta,\gamma]$ contained all

$$f(z) = \frac{1}{z} + \frac{2r\delta\gamma(1-\beta)z}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}} + \sum_{n=2}^{\infty} |a_n|z^{nq}, 0 \leq r \leq 1 \quad (4.\ 11)$$

Theorem 4.2. *Let the functions defined by* (4. 11) *belong to* $\sum_{r,\alpha}^{+}[\beta,\delta,\gamma]$ *iff*

$$\sum_{n=2}^{\infty} \{(nq+1)+\delta\left[(1-nq)+2\gamma(nq-\beta)\right]\} |a_n| \leq 2\delta\gamma(1-\beta)(1-r). \quad (4.\ 12)$$

It is true for

$$f_n(z) = \frac{1}{z} + r\frac{2\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}z + \frac{2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}z^{nq}, n \geq 2.$$

Proof. By using Theorem1.1, we have

$$\sum_{n=1}^{\infty}\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}|a_n| \leq 2\delta\gamma(1-\beta),$$

this implies that

$$\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}\,|a_1|$$

$$+\sum_{n=2}^{\infty}\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}|a_n| \leq 2\delta\gamma(1-\beta),$$

replacing $|a_1|$ by $\frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}$, $0 \leq r \leq 1$, and after doing some mathematics, we proved inequality given in (4. 12). □

Theorem 4.3. *If the functions defined by* (4. 11) *belong to* $\sum_{r,\alpha}^{+}[\beta,\delta,\gamma]$ *then* f *is convex in* $0<|z|<\rho(\beta,\delta,\gamma,r)$ *where* $\rho(\beta,\delta,\gamma,r)$ *is the largest value for which*

$$\frac{3r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}\rho^2+\frac{nq(nq+2)2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}\rho^{nq+1} \leq 1,\ n\Big|_{n=2}^{\infty}.$$

Sharp function for the result is given by

$$f_n(z)=\frac{1}{z}+\frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}z+\frac{2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}z^{nq}.$$

Proof. Here we need to get

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1,$$

for the functions defined by (4. 11). Therefore, we have

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq \frac{\frac{2r\delta\gamma\rho^2}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}+\sum\limits_{n=2}^{\infty}nq(nq+1)|a_n|\rho^{nq+1}}{1-\frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}\rho^2-\sum\limits_{n=2}^{\infty}nq|a_n\rho^{nq+1}} < 1,$$

whenever

$$\frac{3r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}\rho^2+\sum_{n=2}^{\infty}nq(nq+2)|a_n|\rho^{nq+1} < 1.$$

Since $f \in \sum_{r,\alpha}^{+}[\beta,\delta,\gamma]$, hence we may take

$$|a_n|=\frac{2\lambda_n\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}},\quad \sum_{n=2}^{\infty}\lambda_n \leq 1.$$

For each fixed ρ, choose an integer $n=n(\rho)$ for which $\frac{nq(nq+2)\rho^{nq+1}}{\{(nq+1)+\beta[(1-nq)+2\gamma(nq-\beta)]\}}$ is maximal. Then

$$\sum_{n=2}^{\infty}nq(nq+2)|a_n|\rho^{nq+1} \leq \frac{nq(nq+2)2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}\rho^{nq+1},$$

Now find the value $\rho_0=\rho(\beta,\delta,\gamma,r)$ and the corresponding $n(\rho_0)$ so that

$$\frac{3r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}}\rho_0^2+\frac{nq(nq+2)2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}\rho_0^{nq+1}=1.$$

For this value function f is convex in defined range of $|z|$. □

Theorem 4.4. *Let*

$$f_i(z) = \frac{1}{z} + \frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}} z,$$

and

$$f_n(z) = \frac{1}{z} + \frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}} z + \frac{2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}} z^{nq}, \ n|_{n=2}^{\infty}, \tag{4. 13}$$

then $f(z)$ belongs to $\sum_{r,\alpha}^{+}[\beta,\delta,\gamma] \iff f(z) = \sum\limits_{n=0}^{\infty} \lambda_n f_n(z)$, with $\lambda_n \geq 0$ and $\sum\limits_{n=0}^{\infty} \lambda_n =$ 1.

Proof. Suppose that $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, then after some calculation and using hypothesis, we get

$$f_n(z) = \frac{1}{z} + \frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}} z + \sum_{n=2}^{\infty} \frac{2\delta\gamma(1-\beta)(1-r)\lambda_n}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}} z^{nq} \tag{4. 14}$$

by using Theorem 4.2, we have

$$\sum_{n=2}^{\infty} \frac{2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}} \lambda_n \frac{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}{2\delta\gamma(1-\beta)(1-r)}$$

$$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1,$$

therefore $f(z) \in \sum_{r,\alpha}^{+}[\beta,\delta,\gamma]$.
Conversely, we suppose that

$$f(z) = \frac{1}{z} + \frac{2r\delta\gamma(1-\beta)}{\{(q+1)+\delta[(1-q)+2\gamma(q-\beta)]\}} z + \sum_{n=2}^{\infty} |a_n| z^{nq},$$

belonging to class, discussed for (4. 11), and

$$|a_n| \leq \frac{2\delta\gamma(1-\beta)(1-r)}{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}$$

setting as

$$\lambda_n = \frac{\{(nq+1)+\delta[(1-nq)+2\gamma(nq-\beta)]\}}{2\delta\gamma(1-\beta)(1-r)} |a_n|, \ n|_{n=1}^{\infty},$$

and $\lambda_1 = 1 - \sum_{n=1}^{\infty} \lambda_n$. Then clearly $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, as required. □

Other work related to analytic functions and its properties can be found in [6-8].

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